APPM 2360 Project 1 Report
Fixed-Rate Vs Adjustable-Rate Mortgage Analysis

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Abstract

In this report, we compare fixed-rate and adjustable-rate mortgages for financing a home purchase in Boulder, CO, using time-value-of-money growth models and a continuous-time approximation of mortgage amortization. We begin by reviewing discrete versus continuous compounding and quantifying how compounding frequency affects loan growth. We then model the mortgage balance, analyze the equilibrium balance, and derive a closed-form solution for the remaining loan balance over time. From this solution, we obtain an explicit formula for the monthly payment required to pay off a given principal over a specified loan term, and we use it to compare affordability (monthly payment) versus total borrowing cost (total paid and total interest) for 10-year and 30-year mortgage scenarios. We also evaluate the effects of a down payment on the monthly payment and the total interest paid. Finally, we use Euler's method to approximate payoff times for both constant-rate and time-varying-rate models, showing how step size affects numerical accuracy and how rising rates in an adjustable-rate mortgage can extend payoff time for the same monthly payment.

Contents

Background: Mathematical Framework

This report uses standard time-value-of-money growth models and a continuous-time approximation for mortgage amortization. The models below provide a consistent reference for all subsequent calculations and numerical analyses.

Compounding Conventions

Under discrete compounding with n compounding periods per year, an initial balance A_0 grows to

$$A(t) = A(0)\left(1 + \frac{r}{n}\right)^{nt} \tag{1}$$

where r is the annual interest rate converted from a percentage to a decimal, and t is measured in years. $A(t)$ is measured in dollars.

Conversely, a continuously compounded loan grows to

$$A(t) = A_0 e^{rt}. \tag{2}$$

Equilibrium Solutions and Stability

In this section, we introduce the continuous-time amortization model that will be used in the fixed-rate analysis to study long-term behavior and equilibria. It is useful to understand how the differential equation below captures the net change in the loan balance for accruing interest whilst regular payments are made.

$$A' = rA - 12p \quad (r > 0, p > 0) \tag{3}$$

- $A(t)$ denotes the remaining loan ballance in dollars at time t (in years).
- rA models the growth of the balance due to interest: when the annual interest rate is r, the balance increases at a rate proportional to A.
- $12p$ represents the total amount paid toward the loan per year, since p is a monthly payment and there are 12 months in a year.
- Thus the differential equation states that the loan balance changes according to

$$\text{(interest added per year)} - \text{(payments made per year)}.$$

We use this equation assuming that $r > 0, p > 0$ because r represents a positive interest rate, and p represents a positive monthly payment. This ensures that the model represents a continuous amortization where the creditor generates money from providing the loan and that the debtor is making the required monthly payments as per the contract.

Exact Solution To The Amortization Model

In order to compare mortgage options in a consistent way, we model the remaining loan balance as a function of time. Let $A(t)$ be the amount still owed (in dollars) at time t (measured in years). Interest increases the balance at a rate proportional to the current balance, while regular payments decrease the balance at a constant rate. If the annual interest is r (written as a decimal) and the monthly payment is p, then the total paid over a year is $12p$. this leads to the continuous-time amortization model.

$$\text{Rate Of Change Of Balance} = \text{Interest Added Per Year - Payments Made Per Year}$$

We can write this mathematically as:

$$A' = rA - 12p \tag{4}$$

An important reference value is A^*—the equilibrium balance—this is defined as the balance in which the yearly interest exactly equals the yearly payments. When the balance equals A^*, the balance will remain constant over time. Solving for A^* yields

$$A^* = \frac{12p}{r} \tag{5}$$

The value of (5) will be used repeatedly to interpret whether a given payment is large enough to reduce the balance of the loan, or whether the balance will grow.

Monthly Payment Required For Payoff

When considering fixed-rate mortgages, one of the most important qualities is the size of the monthly payment p, since it determines whether the balance decreases fast enough to pay off the loan within a desired time-frame. In our continuous-time amortization model, the remaining balance $A(t)$ depends on the interest rate r, the monthly payment p, and the initial loan amount A_0. Once an exact expression for $A(t)$ is known, we can enforce a payoff requirement by prescribing a mortgage length t_l (in years) and imposing the condition than the balance at that time is zero: $A(t_l) = 0$.
This is represented mathematically as:

$$0 = \frac{12p}{r} + (A_0 - \frac{12p}{r})e^{rt_l} \tag{6}$$

This condition represents the idea that after t_l years of paying off the loan with a monthly amount p, the loan has been retired. Solving the payoff condition analytically for p, produces a closed-form payment formula in terms of A_0, r, and t_l. This formula will be used to compute the required monthly payment required for different fixed-rate mortgage models. Furthermore it will be used to compare how changes in rate and length affect the affordability of the loan (p monthly payments) versus the overall cost of the loan (A total payments and interest).

Once we know what p is, we can solve for the total amount that was paid over the duration of the mortgage:

$$P_{\text{tot}} = 12\,p\,t_l, \tag{7}$$

Where P_{tot} is the total amount paid in dollars, as p is paid 12 times a year for t_l years.

Additionally, we can use P_{tot} to calculate the total interest the loan generated, with the Interest as I_{tot}

$$I_{\text{tot}} = P_{\text{tot}} - A_0, \tag{8}$$

As the total interest is simply the total amount paid minus the initial loan amount.

Effect Of A Down Payment On Monthly Payments And Interest

A down payment reduces the amount that must be borrowed, so in the amortization model it replaces the initial balance A_0 with a smaller principal. Because the payoff-payment formula depends on A_0, lowering A_0 lowers the required monthly payment for any fixed rate r and loan horizon t_l. Beyond affordability, the main financial benefit of a down payment is reduced interest paid over the life of the mortgage: since the total cost of the house is unchanged, the difference in total out-of-pocket cost between the two scenarios is exactly the difference in interest paid. In the sections that follow, we recompute the payoff payments and the total interest for $A_0 = \$650{,}000$ (corresponding to a \$100,000 down payment) using the same rates and horizons as before, and quantify the interest savings relative to the $A_0 = \$750{,}000$ case.

Problem Statement

We consider a prospective home purchase in Boulder, Colorado, and we compare fixed-rate and adjustable-rate mortgage options using analytic and numerical methods from differential equations. Let $A(t)$ denote the remaining loan balance (in dollars) at time t (in years). The goals of this report are:

1. Quantify how the choice of compounding convention (annual, semiannual, quarterly, monthly, and continuous) affects the growth of a loan balance over time for a fixed nominal annual rate.
2. Model mortgage amortization in continuous time with the differential equation $A' = rA - 12p$, where r is the annual interest rate, and p is the monthly payment, and use this model to identify the equilibrium balance, interpret its meaning, and derive a closed-form expression for $A(t)$.
3. Use the closed-form solution to derive a payment formula that guarantees payoff by a specified loan term, and apply it to compare 10-year and 30-year fixed-rate scenarios in terms of monthly affordability and total borrowing cost (total paid and total interest).
4. Evaluate the effect of a down payment by repeating the fixed-rate payment and total-cost computations with a reduced principal, and quantify the resulting changes in monthly payment and total interest.
5. Approximate payoff behavior numerically using Euler's method for both a fixed-rate mortgage and an adjustable-rate mortgage with a time-dependent rate $r(t)$, and examine how step size impacts numerical accuracy and predicted payoff time.

These analyses provide a consistent framework for comparing mortgage options and highlight the tradeoffs between monthly payment affordability, long-run interest cost, and interest-rate risk.

Calculations

Analysis Of Fixed-Rate Mortgage Model

Effect of Continuous Compounding on the Value of a Loan

To compare how different compounding rates affect the total cost of a loan, we will compare the total cost of a loan compounded at different time intervals. Using (1), we compute the balance at $t = 5$ years, with an interest rate of 3% ($r = 0.03$), and the given initial loan value of $A_0 = \$750,000$. We calculate the total value of loans compounded 1, 2, 4, and 12 times per year, with no payments, and compare this value to the value of a loan compounded continuously, whose equation is given by (2). The summary of our results are shown in Table 1.

We visualize the long-term behavior of the loan by plotting the total loan amount over the time interval $0 \le t \le 30$ for quarterly, monthly, and continuous compounding, as shown in Figure 1. All computations and graphs were generated in MATLAB (Listing A.1).

Equilibrium Solutions And Stability

For a fixed-rate payment model we use equation (3) with $r > 0, p > 0$.

An equilibrium solution is a constant balance $A(t) = A^*$ such that $A' = 0$. We set the RHS of the equation to zero and solve for the equilibrium equation: (5) We determine the stability of our solution by rewriting the differential equation in terms of A^*:

$$A^* = r(A - A^*) \tag{9}$$

since $r > 0$,

- If $A > A^*$ then$A' > 0$ and the balance increases
- If $A < A^*$ then $A' < 0$ and the balance decreases

$\therefore$ the solutions move away from A^* so the equilibrium solution is unstable.

A^* is the break-even point where the balance paid $12p$ equals the yearly interest rA.

Exact Solution To The Amortization Model

We determine the exact behavior of the loan balance by solving the amortization model (4) with initial condition $A(0) = A_0$, where r and p are constants. Solving this IVP yields

$$A(t) = \frac{12p}{r} + \left(A_0 - \frac{12p}{r}\right) e^{rt}. \tag{10}$$

The derivation is included in Appendix B.1. To interpret (10), recall the equilibrium balance $A^* = \frac{12p}{r}$ from (5). Subtracting A^* from both sides of (10) gives

$$A(t) - A^* = (A_0 - A^*)\, e^{rt}. \tag{11}$$

Since $r > 0$, the factor e^{rt} increases with t, so solutions move away from A^*. In particular:

- If $A_0 > A^*$, then $A(t)$ increases over time (payments are not large enough to reduce the balance).
- If $A_0 < A^*$, then $A(t)$ decreases and reaches 0 in finite time (the loan is paid off).
- If $A_0 = A^*$, then $A(t) = A^*$ for all $t \ge 0$ payments exactly match interest, and the balance remains constant (The loan will not be paid off, but the balance won't increase).

Monthly Payment Required For Payoff

We use the exact balance formula (3) and impose the payoff condition $A(t_l) = 0$, where t_l is the mortgage payoff condition (in years). Based on the exact balance formula (3) Set t_l, and require payoff (Set equal to 0) The monthly payment p for the loan is shown in Table 2. The full derivation for the calculations can be found at

Long-Term Cost Of Low Monthly Payments

While Table 2 shows that the monthly payments for a longer loan can be significantly lower than a shorter loan, this does not imply a lower total cost. By summing the monthly payments over the loan term, we can find the total payment using (7). Additionally we can use (8) to find the total interest generated by both of our loans, which will allow us to quantify the cost of borrowing. The results of the total payments for our 10-year and 30-year fixed mortgages can be found in Table 3.

Effect Of A Down Payment On Monthly Payments And Interest

With a \$100,000 down payment, the initial loan balance decreases from $A_0 = \$750{,}000$ to $A_0 = \$650{,}000$. Using the payoff-payment formula (14)

We compute the corresponding monthly payments for the same fixed-rate horizons as in the previous section. The calculation for the 10-year mortgage is in B.4.1, and the calculation for the 30-year mortgage is in B.4.2.

Term-Length Tradeoffs: Comparison Metrics

A 10-year versus 30-year fixed-rate mortgage primarily trades monthly affordability for total borrowing cost. Using the amortizing payment expression (14), the required monthly payment is a function of the term length t_l and the interest rate r. To compare term lengths in a way that does not depend on specific numerical inputs, we summarize: the monthly payment $p(r, t_l)$, the total amount paid $P_{\text{tot}}(r, t_l)$ from (7), and the total interest $I_{\text{tot}}(r, t_l)$ from (8). When the interest rate is held fixed, increasing the term length lowers the monthly payment but increases total interest paid because payments are made over a longer horizon. The result of our analysis is found in Table 4

Numerical Solutions

Fixed-Rate Mortgage

To approximate the mortgage balance, we apply Euler's method to the fixed-rate model

$$A' = rA - 12p, \qquad A(0) = A_0,$$

with $A_0 = \$750{,}000$, $r = 0.05$, and $p = \$4000$. Using the update formula

$$A_{n+1} = A_n + h\big(rA_n - 12p\big),$$

the solution is advanced forward in time until the balance first becomes negative, which is interpreted as the payoff time to account for discretization error. The computations were performed in MATLAB (Listing A.2).

Adjustable-Rate Mortgage

We next apply Euler's method to the variable-rate model

$$A' = r(t)A - 12p, \qquad A(0) = A_0,$$

where

$$r(t) = \begin{cases} 0.03, & t \le 5, \\ 0.03 + 0.015\sqrt{t-5}, & t > 5. \end{cases}$$

Using step size $h = 0.01$, the numerical solution is advanced until the balance first becomes negative, which is interpreted as payoff. All computations were performed in MATLAB(Listing A.3).

Results

Analysis Of Fixed Rate Mortgages

Comparing Discretely Compounded Rates To The Continuously Compounding Rate

Compounding convention	Balance after 5 years, $A(5)$ ($)
Annual discrete ($n = 1$)	869,455.56
Semiannual discrete ($n = 2$)	870,405.62
Quarterly discrete ($n = 4$)	870,888.11
Monthly discrete ($n = 12$)	871,212.59
Continuous	871,375.68

Table 1: Modeled balance after five years with $A_0 = 750{,}000$ and $r = 0.03$ under different compounding conventions. Computations generated in MATLAB (Listing A.1)

It is clear that the continuously compounded interest rate grows the fastest, and that the annual rate grows the slowest. This ordering is expected: increasing the compounding frequency credits interest to the balance more often, so interest begins earning interest sooner. In fact, the continuous compounding model (2) is the limiting case of the discrete model (1) as $n \to \infty$, so discrete compounding approaches continuous compounding from below as n increases.

Although the differences are modest over a five-year horizon, they are not zero. For this scenario, switching from annual to continuous compounding increases the modeled balance after five years by approximately $1,920, while switching from monthly to continuous compounding changes the balance by only about $163. Over longer horizons, these small differences accumulate, motivating the plot of the trajectories in Figure 1.

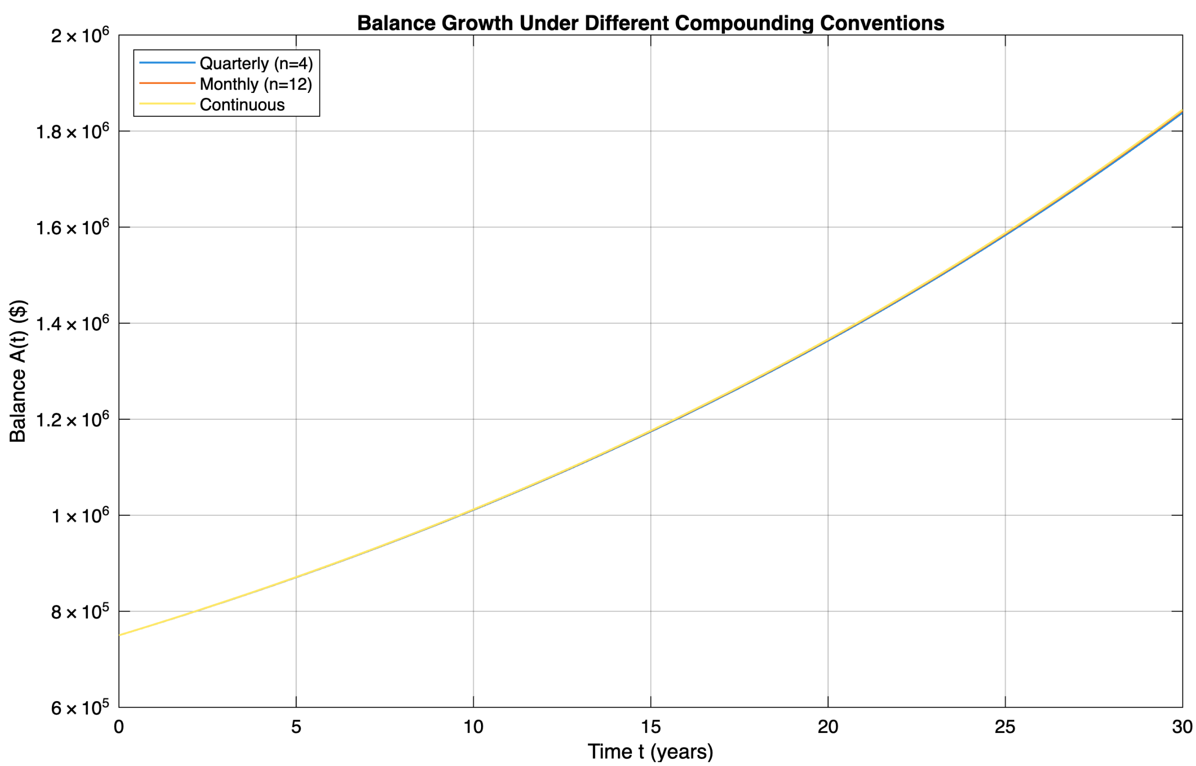

Figure 1: Comparing the quarterly, monthly, and continuously compounding rates of a loan over 30 years. Figure generated in MATLAB (Listing A.1)

Figure 1 shows that the balance trajectories preserve the same ordering observed in Table 1: quarterly discrete compounding grows the slowest, monthly discrete compounding grows slightly faster, and continuous compounding grows the fastest. While the curves appear close together, especially on a linear scale, this is because the compounding convention changes the balance by a small percentage rather than changing the overall exponential trend. Over long horizons, even small percentage differences accumulate into meaningful dollar amounts. By the end of the 30-year window, the continuously compounded balance is separated from the discretely compounded balances by thousands of dollars, illustrating that compounding frequency might not matter much in the short term, but it can make a significant difference in the long term.

Required Monthly Payment For Full Amortization

Using the closed-form payment expression (14), we compute the monthly payment required to fully amortize a principal of $A_0 = \$750{,}000$ by the specified loan horizon. Table 2 reports the required payments for the 10-year and 30-year fixed-rate scenarios.

Mortgage term	Rate r	Horizon t_l (years)	Monthly payment p (\$)
10-year fixed	0.03	10	7,234.30
30-year fixed	0.05	30	4,022.55

Table 2: Monthly Payments Required To fully Amortize A Loan With $A_0 = \$750{,}000$, computed from (14).

The shorter term requires a substantially larger monthly payment because the principal must be repaid over fewer years, even though the interest rate is lower. Conversely, the longer term reduces the monthly payment but spreads repayment over decades, typically increasing the total interest paid; this trade-off is quantified in the total payment and interest calculations that follow, particularly below in Table 3. The full calculations can be found at Appendix B.2

Mortgage term	p (\$/mo)	t_l (yrs)	Total paid P_{tot} (\$)	Total interest I_{tot} (\$)
10-year fixed	7,234.30	10	868,116.58	118,116.58
30-year fixed	4,022.55	30	1,448,119.03	698,119.03

Table 3: Total amount paid and total interest paid for $A_0 = \$750{,}000$, using (7)–(8). Full calculations can be found at Listing B.3

As shown in Table 3, the total cost of the 30-year mortgage is much higher than that of the 10-year mortgage. Specifically, the 30-year mortgage generates almost 7 times the interest as the 10-year mortgage. This confirms our findings that, even though the monthly payment may be lower, the total cost is much higher. In the following section, we compare a general 30-year and 10-year mortgage.

Term-length comparison: affordability versus total cost

Holding the interest rate $r > 0$ fixed and comparing a 30-year term to a 10-year term, the amortizing payment and total-cost expressions imply the directional tradeoffs summarized in Table 4. In other words, extending the term reduces the required monthly payment but increases the total interest paid.

Metric (fixed $r > 0$)	10-year term	30-year term
Monthly payment $p(r, t_l)$	higher	lower
Total paid $P_{\text{tot}}(r, t_l)$	lower	higher
Total interest $I_{\text{tot}}(r, t_l)$	lower	higher
Time to payoff t_l	10 years	30 years

Table 4: Directional comparison of 10-year vs 30-year fixed-rate mortgages when the interest rate is held fixed. The inequalities follow from(7), (8), and (14).

Analysis Of Fixed Rate Mortgage Step Size $h = 0.5$

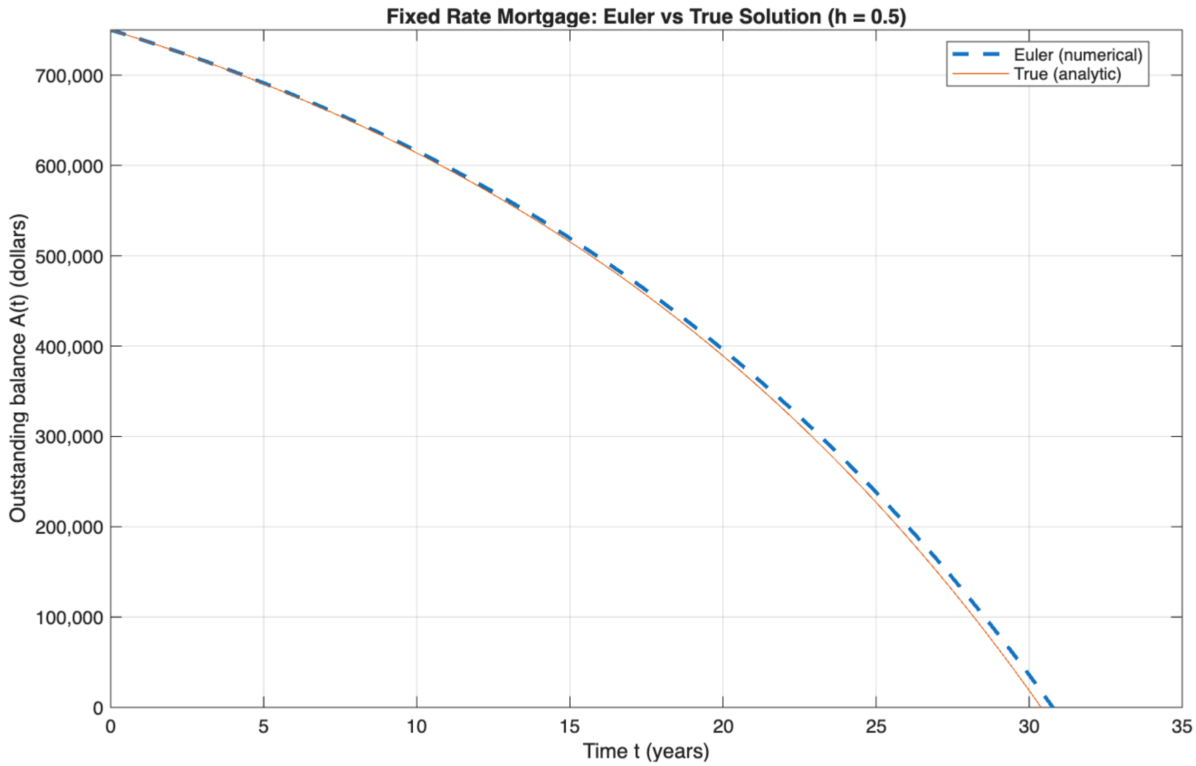

Figure 2: Euler approximation of a fixed-rate mortgage balance with step size $h = 0.5$ compared to the analytic solution. Figure generated in MATLAB (Listing A.2).

Figure 2 compares the Euler approximation with the analytic solution of the mortgage model using a step size of $h = 0.5$ years. The approximation follows the overall behavior of the analytic solution but consistently overestimates the remaining balance. The mortgage is paid off at

$$t = 31.00 \text{ years.}$$

The error accumulates over time and becomes most noticeable near the payoff, where the Euler method predicts a longer payoff time than the analytic solution. This demonstrates that a larger step size reduces the accuracy of Euler's method.

Analysis Of Fixed Rate Mortgage Step-Size $h = 0.01$

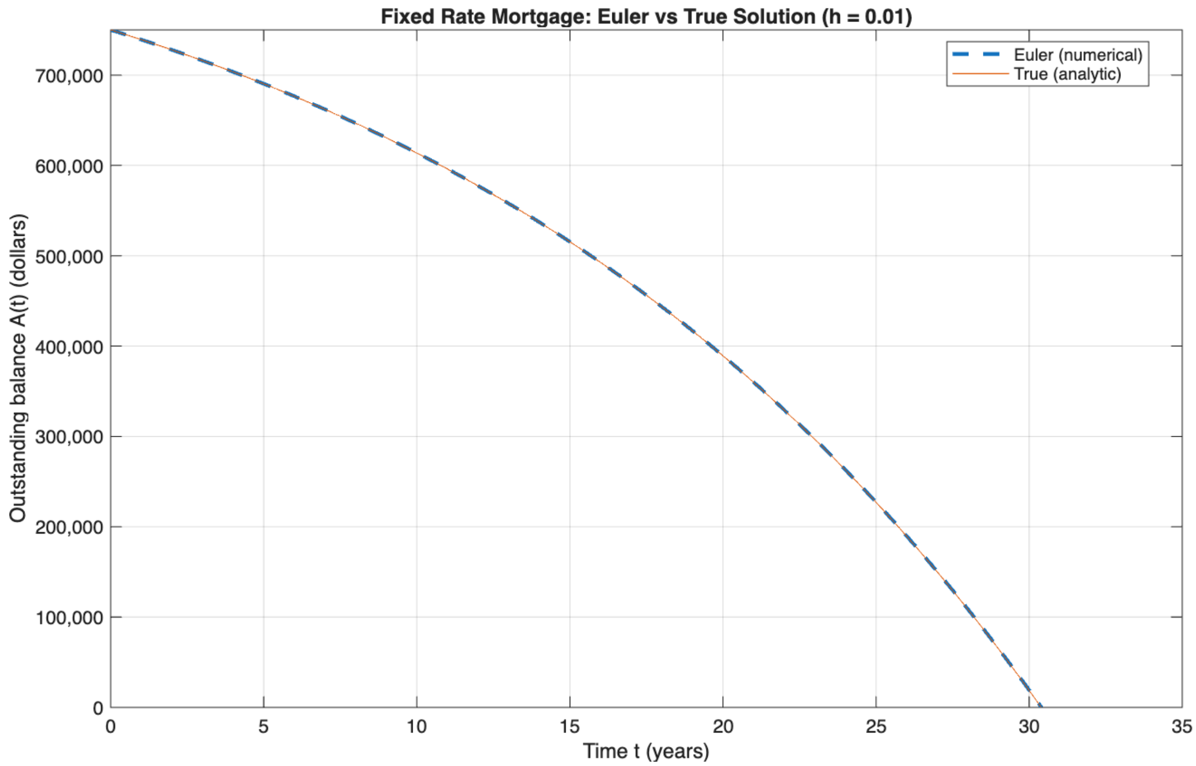

Figure 3: Euler approximation of a fixed-rate mortgage balance with step size $h = 0.01$ compared to the analytic solution. Figure generated in MATLAB (Listing A.2).

Figure 3 shows the Euler approximation with a smaller step size $h = 0.01$ years compared to the analytic solution. The numerical and analytic solutions are nearly indistinguishable over most of the interval, with only a small deviation near the payoff. The mortgage is paid off at

$$t = 30.41 \text{ years.}$$

The predicted payoff time is significantly closer to the true value than in the previous figure, confirming that decreasing the step size improves the accuracy of Euler's method for this specific differential equation.

Adjustable-Rate Mortgage

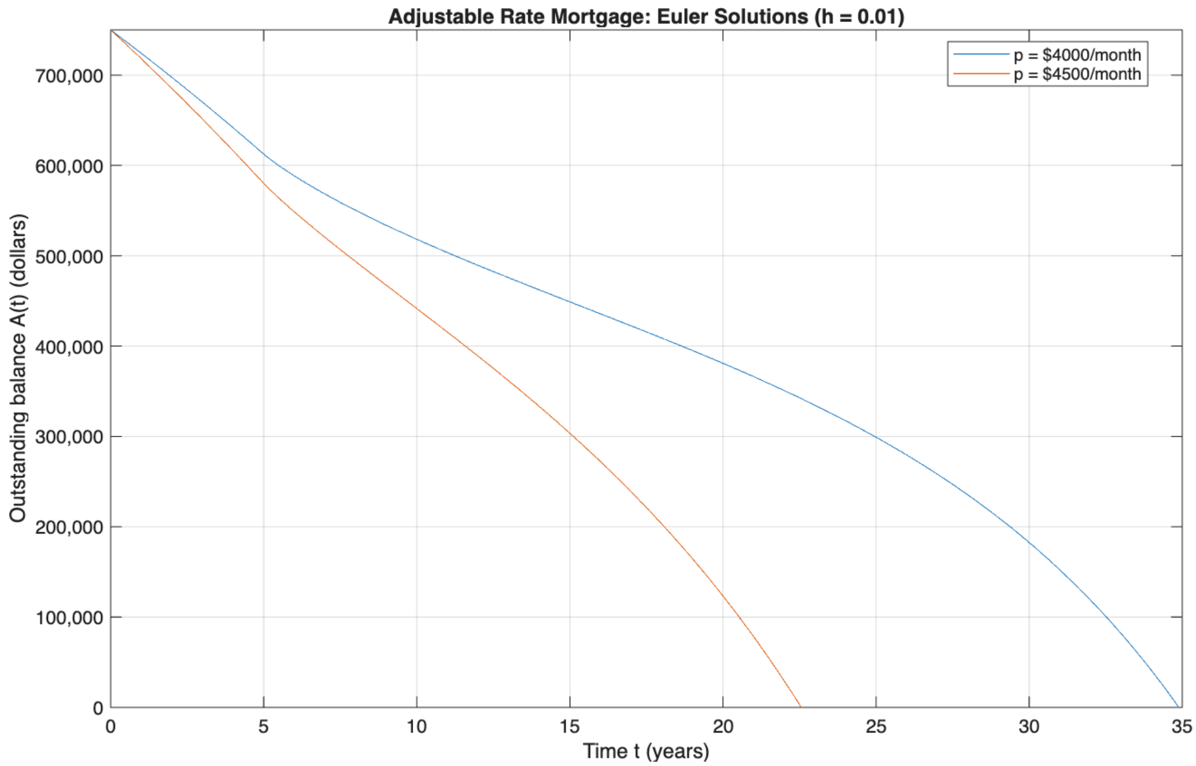

Figure 4: Numerical solution of an adjustable-rate mortgage using Euler's method ($h = 0.01$) for monthly payments of $4000 and $4500. Figure generated in MATLAB (Listing A.3).

Figure 4 shows the mortgage balance for an adjustable-rate loan where the interest rate increases after five years.

For a monthly payment of $4000, the mortgage is paid off at

$$t = 34.90 \text{ years},$$

while for a monthly payment of $4500, it is paid off at

$$t = 22.56 \text{ years}.$$

The balance decreases rapidly at first due to the low initial interest rate, but repayment slows as the rate rises. A larger monthly payment produces a faster reduction of the balance and a shorter payoff time. Compared to the fixed-rate case, a variable interest rate alters the curvature of the solution, illustrating how higher interest rates reduce the effectiveness of each payment over time. For our specific solution, it would take our friends two more years to pay off their mortgage with the same monthly payment, despite having started with a considerably lower interest rate.

Conclusion

Both the analytic and numerical models reach the same conclusion: mortgage decisions are fundamentally a trade-off between short-term affordability and long-term borrowing costs. In the time-value-of-money comparison (Table 1 and Figure 1), increasing compounding frequency produces slightly faster balance growth, with continuous compounding representing the limiting "fastest-growth" case. Although the differences are almost moot in the first couple of years, the separation grows over decades, reinforcing the idea that small rate effects accumulate over long periods.

For fixed-rate mortgages, the continuous-time amortization model provides a clear way to connect monthly payment size to the payoff time and total cost. Using the payoff-payment expression (14), we computed monthly payments required to retire the loan over 10 years versus 30 years (Table 2). The results show that a 30-year term substantially reduces the required monthly payment, improving cash flow flexibility and reducing the risk of payment strain. Additionally, mortgages with interest rates below the currency interest rate will not grow as quickly as the currency, making them very wise investments. However, the total cost calculations (Table 3) demonstrate the opposing effect: the longer horizon leads to dramatically higher total interest paid, so a lower monthly payment does not imply a cheaper loan overall. The directional comparison in Table 4 summarizes this general tradeoff: holding the rate fixed, longer terms lower p but increase both total paid and total interest.

The effect of a down payment follows directly from the same payoff-payment and total-cost formulas: reducing the borrowed principal lowers the required payment and total interest paid. In our computations with a \$100,000 down payment ($A_0 = \$650{,}000$), the reduction in interest paid is meaningful in both terms and especially pronounced for the longer mortgage, because interest accrues over many more years. This makes down payments a powerful lever for both affordability and cost minimization, particularly when combined with a longer-term mortgage.

Finally, the numerical results underscore why adjustable-rate mortgages (ARMs) require caution when future interest rates are uncertain. Euler's method approximations show that step size affects numerical accuracy (Figures 2 and 3), with smaller steps producing payoff times much closer to the analytic solution. In the adjustable-rate model (Figure 4), increasing interest rates after five years can slow repayment for the same monthly payment and extend the payoff horizon by years. As a result, ARMs can be attractive if borrowers expect to move or refinance before rate increases dominate (i.e., they expect to spend a shorter time in the house), but they become riskier for long-term horizons because rising rates can increase total cost and lengthen the payoff time unless payments increase correspondingly.

Overall, mortgages must be assigned on a case-by-case basis. The most practical recommendation depends on how long our friends expect to remain in the home. If they expect to stay long-term and can afford the larger monthly payment, the shorter-term fixed-rate mortgage minimizes total interest and builds equity faster. If monthly affordability and flexibility are the priorities, a longer-term fixed-rate mortgage lowers the monthly payment requirement, but comes with a substantially higher lifetime cost. If they expect to move or refinance within a relatively short time frame, an ARM may be reasonable, but only with a clear plan for the rate adjustment period and sufficient financial stability to handle higher future rates.

A MATLAB Scripts

A.1 3.1.1 - Compounding Comparison

```
%% Compounding Comparison (Section 3.1.1)
% This script compares continuous and discrete compounding for a loan with
% no payments made. It:
%   - Computes A(5) under discrete compounding for n = 1,2,4,12
%   - Computes A(5) under continuous compounding
%   - Prints a summary table to the Command Window (for LaTeX table entry)
%   - Plots A(t) for t in [0,30] for quarterly, monthly, and continuous
%   - Saves the plot to figures/sec311_compounding.png

clear; clc; close all;

%% Parameters (scenario inputs)
A0 = 750000;      % initial balance ($)
r  = 0.03;        % nominal annual interest rate (decimal)

t_eval = 5;       % evaluation time in years
n_list = [1 2 4 12];

%% Discrete compounding model: A(t) = A0 * (1 + r/n)^(n*t)
A5_discrete = zeros(size(n_list));
for k = 1:numel(n_list)
    n = n_list(k);
    A5_discrete(k) = A0 * (1 + r/n)^(n * t_eval);
end

%% Continuous compounding model: A(t) = A0 * exp(r*t)
A5_continuous = A0 * exp(r * t_eval);

%% Print results to Command Window (copy into LaTeX table)
fprintf('\nSection 3.1.1: Balance after %.0f years (A0=%.0f, r=%.2f%%)\n', ...
        t_eval, A0, 100*r);
fprintf('------------------------------------------------------------\n');
fprintf('%-28s %-8s %15s\n', 'Convention', 'n', 'A(5) [$]');
fprintf('------------------------------------------------------------\n');

for k = 1:numel(n_list)
    fprintf('%-28s %-8d %15.2f\n', 'Discrete compounding', n_list(k),
        A5_discrete(k));
end
fprintf('%-28s %-8s %15.2f\n', 'Continuous compounding', '-', A5_continuous);

fprintf('------------------------------------------------------------\n\n');

%% Plot over 0 <= t <= 30 years (quarterly, monthly, continuous)
t = linspace(0, 30, 1000);

n_quarter = 4;
n_month   = 12;

A_quarter = A0 * (1 + r/n_quarter).^(n_quarter * t);
A_month   = A0 * (1 + r/n_month).^(n_month * t);
A_cont    = A0 * exp(r * t);

figure;
```

```
plot(t, A_quarter, 'LineWidth', 2); hold on;
plot(t, A_month,   'LineWidth', 2); hold on;
plot(t, A_cont,    'LineWidth', 2);
grid on;

xlabel('Time t (years)');
ylabel('Balance A(t) ($)');
title('Balance Growth Under Different Compounding Conventions');
legend('Quarterly (n=4)', 'Monthly (n=12)', 'Continuous', 'Location',
    'northwest');

ax = gca; ax.YAxis.Exponent = 0;

%% Save plot for LaTeX
if ~exist('figures', 'dir')
    mkdir('figures');
end

outFile = fullfile('figures', 'sec311_compounding.png');
exportgraphics(gcf, outFile, 'Resolution', 300);

fprintf('Saved figure to: %s\n\n', outFile);
```

A.2 3.2.1 - Fixed Rate Mortgage

```
%% Project 1 APPM 2360 - Section 3.2 Numerical Solutions (Euler's Method)

clear;                          % Clear workspace variables
clc;                            % Clear command window
close all;                      % Close all figures

%% Common given values

initial_debt = 750000;          % A0 in dollars
annual_rate_fixed = 0.05;       % Fixed rate r for 3.2.1
monthly_payment_fixed = 4000;   % Payment p for 3.2.1 (dollars per month)

%% 3.2.1 Fixed rate mortgage (Euler + true solution)
% ODE: A' = r*A - 12*p
% Euler: A_{n+1} = A_n + h*(r*A_n - 12*p)
% Paid off when A first becomes negative

% Put the two step sizes in a list so we can run the same code twice
step_sizes = [0.5, 0.01];       % h in years (given)

for k = 1:length(step_sizes)    % Loop over h values
    h = step_sizes(k);          % Current step size

    t_values = 0;               % Start time vector with t0 = 0
    debt_values = initial_debt; % Start debt vector with A0

    current_time = 0;           % Current time (years)
    current_debt = initial_debt;% Current debt (dollars)

    while current_debt >= 0     % Stop when debt first becomes negative
        slope = annual_rate_fixed*current_debt - 12*monthly_payment_fixed; %
            f(t,A)
        next_debt = current_debt + h*slope;                               %
            Euler update
        next_time = current_time + h;                                     % Next
            time

        t_values(end+1,1) = next_time;     % Append time
        debt_values(end+1,1) = next_debt;  % Append debt

        current_time = next_time;          % Update current time
        current_debt = next_debt;          % Update current debt
    end

    payoff_time = t_values(end);           % Time when it first became negative
    payoff_time = round(payoff_time, 2);   % Round to nearest hundredth of a year

    % True (analytic) solution for constant r and constant p:
    % A(t) = (A0 - 12p/r)*exp(r t) + 12p/r
    true_debt_values = (initial_debt -
        (12*monthly_payment_fixed)/annual_rate_fixed) ...
                       .* exp(annual_rate_fixed.*t_values) ...
                       + (12*monthly_payment_fixed)/annual_rate_fixed;      %
                           True solution

    fprintf('3.2.1 Fixed rate: h = %.2f years -> payoff time = %.2f years\n', h,
        payoff_time);
```

```

    figure;                                       % Make a new figure
    plot(t_values, debt_values, '--',LineWidth=2);          % Plot Euler solution
    hold on;                                      % Keep same axes
    plot(t_values, true_debt_values, '-');       % Plot true solution
    hold off;                                     % Release axes

    xlabel('Time t (years)');                     % x-axis label
    ylabel('Outstanding balance A(t) (dollars)');% y-axis label
    title(['Fixed Rate Mortgage: Euler vs True Solution (h = ', num2str(h), ')']); % Title
    legend('Euler (numerical)', 'True (analytic)', 'Location', 'best');    % Legend
    grid on;                                      % Add grid
    ax = gca;                   % get current axes
    ax.YAxis.Exponent = 0;      % remove the ×10^n scaling
    ytickformat('%,.0f');       % commas, no decimals
    ylim([0 initial_debt])

end
```

A.3 3.2.2 - Adjustable Rate Mortgage

```
%% Project 1 APPM 2360 - Section 3.2 Numerical Solutions (Euler's Method)

clear;                          % Clear workspace variables
clc;                            % Clear command window
close all;                      % Close all figures

%% Common given values

initial_debt = 750000;          % A0 in dollars
annual_rate_fixed = 0.05;       % Fixed rate r for 3.2.1
monthly_payment_fixed = 4000;   % Payment p for 3.2.1 (dollars per month)

%% 3.2.2 Adjustable rate mortgage (Euler only, h = 0.01)
% ODE: A' = r(t)*A - 12*p
% r(t) = 0.03 for t <= 5
% r(t) = 0.03 + 0.015*sqrt(t - 5) for t > 5
% Use Euler with h = 0.01, and do p = 4000 and p = 4500
% Compute payoff time and interest paid in each case

h_arm = 0.01;                               % Step size for ARM (given)
payment_options = [4000, 4500];             % Two monthly payments to test

arm_results_time = zeros(size(payment_options));      % Store payoff times
arm_results_interest = zeros(size(payment_options));  % Store interest paid

arm_time_arrays = cell(size(payment_options));        % Store time arrays for plotting
arm_debt_arrays = cell(size(payment_options));        % Store debt arrays for plotting

for j = 1:length(payment_options)           % Loop over both payment scenarios
    monthly_payment_arm = payment_options(j); % Current payment p

    t_values = 0;                           % Start time list
    debt_values = initial_debt;             % Start debt list

    current_time = 0;                       % Current time
    current_debt = initial_debt;            % Current debt

    while current_debt >= 0                 % Stop when debt first becomes negative

        if current_time <= 5                % Piecewise interest definition
            annual_rate = 0.03;             % r(t) for t <= 5
        else
            annual_rate = 0.03 + 0.015*sqrt(current_time - 5); % r(t) for t > 5
        end

        slope = annual_rate*current_debt - 12*monthly_payment_arm; % f(t,A)
        next_debt = current_debt + h_arm*slope;                    % Euler update
        next_time = current_time + h_arm;                          % Next time

        t_values(end+1,1) = next_time;      % Append time
        debt_values(end+1,1) = next_debt;   % Append debt

        current_time = next_time;           % Update time
        current_debt = next_debt;           % Update debt
```

```
    end

    payoff_time = t_values(end);           % Time when debt first became negative
    payoff_time = round(payoff_time, 2);   % Round time to 0.01 year

    % Interest paid = (total paid) - principal
    % Total paid = monthly_payment * (number of months)
    % We approximate months by: months = 12 * payoff_time (years -> months)
    months_paid = 12*payoff_time;                          % Convert years to months
    total_paid = monthly_payment_arm * months_paid;        % Total dollars paid
    interest_paid = total_paid - initial_debt;             % Interest dollars paid
    interest_paid = round(interest_paid, 2);               % Round to cents

    arm_results_time(j) = payoff_time;                     % Store payoff time
    arm_results_interest(j) = interest_paid;               % Store interest paid
    arm_time_arrays{j} = t_values;                         % Store time curve
    arm_debt_arrays{j} = debt_values;                      % Store debt curve

    fprintf('3.2.2 ARM: p = $%d/month -> payoff time = %.2f years, interest paid = $%.2f\n', ...
            monthly_payment_arm, payoff_time, interest_paid);
end

% Plot both ARM scenarios on the same graph
figure;                                                    % New figure
plot(arm_time_arrays{1}, arm_debt_arrays{1}, '-');        % Plot p=4000 case
hold on;                                                  % Keep axes
plot(arm_time_arrays{2}, arm_debt_arrays{2}, '-');        % Plot p=4500 case
hold off;                                                 % Release axes

xlabel('Time t (years)');                                 % x-axis label
ylabel('Outstanding balance A(t) (dollars)');              % y-axis label
title('Adjustable Rate Mortgage: Euler Solutions (h = 0.01)'); % Title
legend('p = $4000/month', 'p = $4500/month', 'Location', 'best'); % Legend
grid on;                                                  % Grid on
ax = gca;                  % get current axes
ax.YAxis.Exponent = 0;     % remove the ×10^n scaling
ytickformat('%,.0f');      % commas, no decimals
ylim([0 initial_debt])
```

B Full Calculations

B.1 Derivation Of The Exact Solution to the Amortization Model

We derive the closed-form solution of the amortization model

$$A' = rA - 12p, \qquad A(0) = A_0. \tag{12}$$

where r and p are constants. Rewrite (12) in standard linear form:

$$A' - rA = -12p.$$

This is a first-order linear ODE with integrating factor

$$\mu(t) = e^{\int -r\,dt} = e^{-rt}.$$

Multiply both sides by $\mu(t)$:

$$e^{-rt}A' - re^{-rt}A = -12p\,e^{-rt}.$$

Observe that the left-hand side is the derivative of a product:

$$\frac{d}{dt}\big(e^{-rt}A(t)\big) = -12p\,e^{-rt}.$$

Integrate both sides with respect to t:

$$e^{-rt}A(t) = \int -12p\,e^{-rt}\,dt + C.$$

Since p and r are constants,

$$\int -12p\,e^{-rt}\,dt = -12p\left(\frac{-1}{r}e^{-rt}\right) = \frac{12p}{r}e^{-rt}.$$

Substitute back:

$$e^{-rt}A(t) = \frac{12p}{r}e^{-rt} + C.$$

Multiply by e^{rt} to solve for $A(t)$:

$$A(t) = \frac{12p}{r} + Ce^{rt}.$$

Apply the initial condition $A(0) = A_0$:

$$A_0 = \frac{12p}{r} + C \quad \Rightarrow \quad C = A_0 - \frac{12p}{r}.$$

Therefore,

$$A(t) = \frac{12p}{r} + \left(A_0 - \frac{12p}{r}\right)e^{rt}. \tag{13}$$

Finally, using $A^* = \frac{12p}{r}$ from (5), we can rewrite (13) as

$$A(t) - A^* = (A_0 - A^*)e^{rt},$$

which is the form used to interpret the behavior of the balance relative to the equilibrium.

B.2 Payoff Payment Computations

Imposing the payoff condition $A(t_l) = 0$ on the exact balance solution

$$A(t) = \frac{12p}{r} + \left(A_0 - \frac{12p}{r}\right) e^{rt}$$

gives

$$0 = \frac{12p}{r} + \left(A_0 - \frac{12p}{r}\right) e^{rt_l}.$$

Multiplying by r and solving for p yields the closed-form monthly payment formula

$$p = \frac{rA_0}{12\left(1 - e^{-rt_l}\right)}. \tag{14}$$

We now evaluate (14) for the two fixed-rate scenarios for the amortization models that our friends can use.

Numerical Evaluation For $A_0 = \$750{,}000$

Calculations for $r = 0.05$, $t_l = 30$

For a 10-year term at 3% ($r = 0.03$, $t_l = 10$), we have $rt_l = 0.3$ and

$$p = \frac{(0.03)(750{,}000)}{12\left(1 - e^{-0.3}\right)}.$$

Compute the components:

$$(0.03)(750{,}000) = 22{,}500, \qquad e^{-0.3} \approx 0.7408182207,$$

$$1 - e^{-0.3} \approx 0.2591817793, \qquad 12(1 - e^{-0.3}) \approx 3.1101813518.$$

Therefore,

$$p \approx \frac{22{,}500}{3.1101813518} \approx 7234.3048378 \approx 7234.30.$$

Calculations For $r = 0.05$, $t_l = 30$

For a 30-year term at 5% ($r = 0.05$, $t_l = 30$), we have $rt_l = 1.5$ and

$$p = \frac{(0.05)(750{,}000)}{12\left(1 - e^{-1.5}\right)}.$$

Compute the components:

$$(0.05)(750{,}000) = 37{,}500, \qquad e^{-1.5} \approx 0.2231301601,$$

$$1 - e^{-1.5} \approx 0.7768698399, \qquad 12(1 - e^{-1.5}) \approx 9.3224380782.$$

Therefore,

$$p \approx \frac{37{,}500}{9.3224380782} \approx 4022.5528650 \approx 4022.55.$$

B.3 Total Cost Calculation

For the 10-year fixed-rate mortgage with $p = \$7{,}234.30$ and $t_l = 10$,

$$P_{\text{tot}} = 12\,p\,t_l = 12(7{,}234.30)(10) = 868{,}116.58,$$

and therefore

$$I_{\text{tot}} = P_{\text{tot}} - A_0 = 868{,}116.58 - 750{,}000 = 118{,}116.58.$$

For the 30-year fixed-rate mortgage with $p = \$4{,}022.55$ and $t_l = 30$,

$$P_{\text{tot}} = 12\,p\,t_l = 12(4{,}022.55)(30) = 1{,}448{,}119.03,$$

and therefore

$$I_{\text{tot}} = P_{\text{tot}} - A_0 = 1{,}448{,}119.03 - 750{,}000 = 698{,}119.03.$$

B.4 Effect Of A Down Payment On Monthly Payments And Interest

B.4.1 10-Year, 3% Fixed-Rate

For the 10-year, 3% fixed-rate scenario ($r = 0.03$, $t_l = 10$),

$$p = \frac{(0.03)(650{,}000)}{12\left(1 - e^{-(0.03)(10)}\right)} \approx 6269.73,$$

so the required monthly payment is \$6,269.73. The total amount paid over t_l years is $12pt_l$, so

$$12pt_l \approx 12(6269.73)(10) = 752{,}367.60.$$

Therefore the total interest paid is

$$I = 12pt_l - A_0 \approx 752{,}367.60 - 650{,}000 = 102{,}367.60.$$

For comparison, in the corresponding $A_0 = \$750{,}000$ case (Table 2), $p = \$7{,}234.30$ and

$$I_{\$750k} = 12(7234.30)(10) - 750{,}000 = 118{,}116.00.$$

Thus the down payment reduces total interest by

$$I_{\$750k} - I_{\$650k} \approx 118{,}116.00 - 102{,}367.60 = \$15{,}748.40.$$

B.4.2 30-Year, 5% Fixed-Rate

For the 30-year, 5% fixed-rate scenario ($r = 0.05$, $t_l = 30$),

$$p = \frac{(0.05)(650{,}000)}{12\left(1 - e^{-(0.05)(30)}\right)} \approx 3486.21,$$

so the required monthly payment is \$3,486.21. The total amount paid over t_l years is

$$12pt_l \approx 12(3486.21)(30) = 1{,}255{,}035.60.$$

Therefore the total interest paid is

$$I = 12pt_l - A_0 \approx 1{,}255{,}035.60 - 650{,}000 = 605{,}035.60.$$

For comparison, in the corresponding $A_0 = \$750{,}000$ case (Table 2), $p = \$4{,}022.55$ and

$$I_{\$750k} = 12(4022.55)(30) - 750{,}000 = 698{,}118.00.$$

Thus the down payment reduces total interest by

$$I_{\$750k} - I_{\$650k} \approx 698{,}118.00 - 605{,}035.60 = \$93{,}082.40.$$